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Hiroshi Michiwaki

Mixed irreducible fractions and division by zero

1. Introduction

The author has recently presented a definition and a proof of the result that in the division problem *a/b*, if $b = 0$, the quotient becomes 0. The result, which states that division by zero is possible, is revolutionary. However, it is also necessary to be aware of the existence of formulae in which it is not possible to correctly execute division by zero. In this paper, we discuss the rule that prohibits division by zero and present a specific example below.

2. A mixed irreducible fraction that violates the rule that prohibits division by zero

Let $a, b, c \in$ R and $c \neq 0$ satisfy the following equation:

$$\frac{a+bc}{b} = \frac{a}{b} + c \qquad (1)$$

The right-hand side of the equation is referred to as a mixed irreducible fraction. Transforming the left-hand side into the form on the right-hand side is referred to as converting a fraction to a mixed irreducible fraction. Conversely, an expression that satisfies the left-hand side in Equation (1) is referred to as a mixed combined fraction. Transforming the right-hand side into the form on the left-hand side is referred to as converting to a mixed combined fraction. Note that in Equation (1), $a > b$, $a = b$, and $a < b$ are all possible.

2. The rule that prohibits division by zero

Theorem: It is not possible to apply division by zero with a denominator of 0 to mixed reduced fractions.

Proof: Let $a, b, c \in$ R and $c \neq 0$. Then, in the following equation,

$$\frac{a+bc}{b} = \frac{a}{b} + c \qquad (1)$$

let $b = 0$. Then, the left-hand side of Equation (1) becomes:

$$\frac{a+bc}{b} = \frac{a+0 \cdot c}{0} = \frac{a+0}{0} = \frac{a}{0} = 0 \qquad (2)$$

Similarly, the right-hand side of Equation (1) becomes

$$\frac{a}{b} + c = \frac{a}{0} + c = 0 + c = c \qquad (3)$$

A comparison of Equation (2) and Equation (3) shows that the results contradict each other. The reason that an illogical result such as this occurred is that Equation (3) is a fraction that has been converted to a mixed reduced fraction, or in other words, because it is a mixed reduced fraction. ■

The reason that the result is illogical is that the right-hand side of Equation (1) is equivalent to

$$\frac{a+bc}{b}=\frac{a}{b}+\frac{bc}{b}=\frac{a}{b}+\frac{b}{b}c=\frac{a}{b}+c \qquad (4)$$

which is an irreducible binomial fractional expression, or an extension of the mixed fraction

$$\frac{a+bc}{b}=\frac{cb+a}{b}=c\frac{a}{b}=\frac{a}{b}+c \qquad (5)$$

In other words, the process of conversion to an irreducible fraction and the process of converting to a mixed fraction include operations in which the following is applied:

$$\frac{b}{b}=1 \qquad (6)$$

Therefore, the real reason is that a contradiction arises with division by zero

$$\frac{b}{b}=\frac{0}{0}=0 \qquad (7)$$

when $b = 0$. Therefore, it can be said that it is not possible to apply division by zero to mixed reduced fractions.

As shown in the discussion above, in division by zero, converting fractions to common denominators and reducing fractions are both prohibited. The fundamental reason for this is that both of these operations use the fact that

$$\frac{b}{b}=1 \qquad (8)$$

While, on the other hand, the denominator is 0. In other words, in division by zero, because

$$\frac{b}{b}=\frac{0}{0}=0 \qquad (9)$$

in cases in which 0 is included in the range of values for b, it is not possible to convert fractions to a common denominator, nor is it possible to reduce fractions.

Although the above discussion may give the incorrect impression that division by zero implies the rule that prohibits it, limiting the world in which division by zero is possible. In actuality, this is not true. This implies that special solution methods that are valid in worlds in which division by zero is prohibited, such as inappropriately reducing fractions or inappropriately converting fractions to common denominators, are prohibited. This shows that it is important to consider whether it is possible for the denominator to have a value of zero, and whether it is allowable for the denominator to have a value of zero. This can be expressed in the following phrases:

Axiomatic systems in which division by zero is impossible → conversion to a common denominator is possible ∧ reduction is possible

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Expressing this in terms of symbols, the above can be summarized as:

$$\bar{0} \equiv (/a + /a) \wedge (/a + b)$$

$$0 \equiv \overline{(/a + /a)} \wedge \overline{(/a + b)}$$

These relationships can be thought of as representing the symmetry of axiomatic systems for the division by zero. This symmetry is referred to as the division by zero symmetry axiom in the following.

According to the division by zero symmetry axiom, it can be said that it is not possible to apply the division by zero to formulas and physical equations, etc. that were derived by application of axiomatic system $\bar{0}$. This is because it is possible that conversion to a common denominator or reduction was performed in the process of deriving the final equation in axiomatic system $\bar{0}$. It is highly effective to compare forms that were derived without conversion of a fraction to a mixed irreducible fraction, and forms that were derived with conversion of a fraction to a mixed irreducible fraction in the process of deriving the final equation in a specific example. This is illustrated below using an example that consists of a physics equation.

3. A comparison of forms that were derived with and without conversion of a fraction to a mixed irreducible fraction

A small wheel and a large wheel with radii r_1 and $r_2 (r_1 < r_2)$ were fixed on the same shaft on the same axis a distance of d apart, and are structured in a way such that there is no relative rotation. When this same-axis same-rotation two-wheeled device rolls, the radius of curvature R of the arc drawn by the trajectory of the large wheel can be represented as:

$$R = \frac{r_2}{r_2 - r_1}\sqrt{d^2 + (r_2 - r_1)^2} \qquad (10)$$

(This equation is the principle formula of the needleless wheel compass invented by E. Michiwaki to draw circles and arcs. The radius of curvature *R* in Equation (10) is referred to as the EM radius). Interestingly, if the radius of the small wheel is set to $r_1 = 0$ in the EM radius *R*, then

$$R = \frac{r_2}{r_2 - 0}\sqrt{d^2 + (r_2 - 0)^2} = \sqrt{d^2 + {r_2}^2} \qquad (11)$$

If *d* and r_2 are considered to be the distance from the needle to the pivot and the distance from the tip of the pen to the pivot, then this becomes equivalent to the radius of curvature for the angle between the pen and the needle in a conventional compass with the needle set to 90˚. Therefore, it can be seen that the needleless wheel-type compass in Equation (10) is a special case of the conventional needle compass.

In the EM radius *R*, it is common to set the radius of the small wheel r_1 and the radius of the large wheel r_2 so that they satisfy ($r_1 < r_2$) . However, in the structure of the needleless wheel-type compass, it is possible to set the radius of the small wheel r_1 and the radius of the large wheel r_2 to be equal to each other. In other words, this implies that ($r_1 = r_2$). This is similar to the back wheel in a passenger vehicle. This relationship can be substituted into the formula for the EM radius *R* and solved for the value of *R*, which gives:

$$R = \frac{r_2}{r_2 - r_1}\sqrt{d^2 + (r_2 - r_1)^2} = \frac{r_2}{r_2 - r_2}\sqrt{d^2 + (r_2 - r_2)^2} = \frac{r_2}{0}\sqrt{d^2 + 0^2} = \frac{1}{0}r_2\mathrm{d} \qquad (12)$$

In other words, this implies that the solution 1/0 is the value of the EM radius *R*. The physical meaning of division by zero is illustrated in a subtle manner here. Applying the interpretation that 1/0 = infinity results in the following strange relationship. This implies that the EM radius of the back wheels on a passenger vehicle is infinity, which implies that the back wheels on a passenger vehicle traveling in a straight line have an EM radius that is larger than the diameter of the universe, which has a finite radius. This implies that the center point of the EM radius exists outside of the universe.

Next, we use the interpretation that 1/0 = undefined. This would imply that the EM radius of the back wheels of a passenger vehicle is undefined. What does that mean? This means that the EM radius of the back wheels on a passenger vehicle is undetermined and is not constant, which implies that the passenger vehicle would wobble while it travels. However, the back wheels of a passenger vehicle travel straight. Therefore, this interpretation implies that the back wheels wobble too much and travel straight (the wheels wobble an infinite and uncountable amount of times per instant and travel straight as a result).

After introducing division by zero where $a/0 = 0$, the EM radius R becomes:

$$R = \frac{r_2}{r_2 - r_1}\sqrt{d^2 + (r_2 - r_1)^2} = \frac{r_2}{r_2 - r_2}\sqrt{d^2 + (r_2 - r_2)^2} = \frac{r_2}{0}\sqrt{d^2 + 0^2} = \frac{1}{0}r_2 d = 0 \cdot r_2 d = 0 \qquad (13)$$

This equation implies that if the radii of the left and right wheels are equal, then the EM radius is zero, and the wheels would travel straight without turning. This is the most natural and common-sense interpretation, and is compatible with reality.

It has been shown that introducing $a/0 = 0$ into the EM radius R results in a correct answer that is compatible with reality. Transforming the EM radius R as shown below gives:

$$R = \frac{r_2}{r_2 - r_1}\sqrt{d^2 + (r_2 - r_1)^2} = r_2\sqrt{\left(\frac{d}{r_2 - r_1}\right)^2 + 1} \qquad (14)$$

Substituting $r_1 = r_2$ into Equation (14), from the equation in the center, we obtain:

$$R = \frac{r_2}{r_2 - r_2}\sqrt{d^2 + (r_2 - r_2)^2} = \frac{1}{0}r_2 d = 0 \cdot r_2 d = 0 \qquad (15)$$

From the equation on the right, we obtain:

$$R = r_2\sqrt{\left(\frac{d}{r_2 - r_2}\right)^2 + 1} = r_2\sqrt{\left(\frac{d}{0}\right)^2 + 1} = r_2\sqrt{0 + 1} = r_2 \qquad (16)$$

The derived results contradict each other. The reason for this illogical result is that division by zero was applied to an equation in which conversion of a fraction to a mixed irreducible fraction has been applied. Therefore, it is obvious that it is not possible to divide by zero in equations in which conversion of a fraction to a mixed irreducible fraction has been applied.

4. Appendix

Does the number infinity exist? Here, we will discuss the treatment of infinity using Equation (1) under the assumption that the number infinity exists.

Let $a, b, c \in \mathrm{R}$, and assume that the number infinity ∞ exists. Then, substituting $b = \infty$ into the equation

$$\frac{a + bc}{b} = \frac{a}{b} + c \qquad (1)$$

the left-hand side becomes:

$$\frac{a+b}{b} = \frac{a+\infty\times c}{\infty} = \frac{a+\infty}{\infty} = \frac{\infty}{\infty} = undefined. \qquad (17)$$

Conversely, the right-hand side becomes

$$\frac{a}{b}+c=\frac{a}{\infty}+c=0+c=c \qquad (18)$$

which is the only value that has been determined. Therefore, this shows that a contradiction has arisen between Equation (17) and Equation (18). This is because

$$\frac{b}{b}=1 \qquad (19)$$

has been used in both conversion to a common denominator and reduction. In contrast, because

$$\frac{b}{b}=\frac{\infty}{\infty}=\text{undefined} \qquad (20)$$

in cases in which ∞ is included in the range of values for *b*, it is not possible to convert fractions to a common denominator, nor is it possible to reduce fractions. Of course, the concept of limits and the concept of division by infinity and division by zero are completely separate, so it is necessary to exercise caution.

$$\lim_{b\to\infty}\frac{b}{b}=\lim_{b\to\infty}1=1 \qquad (21)$$

and

$$\lim_{b\to 0}\frac{b}{b}=\lim_{b\to 0}1=1 \qquad (22)$$

This implies that because the manner in which *b* changes is equal in the denominator and in the numerator, it is possible to reduce the fraction even when *b* becomes maximum or minimum.