

## 初等関数の級数展開とゼロ除算

## 1. 正弦関数

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots \quad (1)$$

ここで,  $z=1/w$  とすると,

$$\sin \frac{1}{w} = \frac{1}{w} - \frac{1}{3!w^3} + \frac{1}{5!w^5} - \cdots + (-1)^n \frac{1}{(2n+1)!w^{2n+1}} + \cdots \quad (2)$$

であるから,  $w=0$  を代入すると,

$$\begin{aligned} \sin \frac{1}{0} &= \frac{1}{0} - \frac{1}{3! \cdot 0^3} + \frac{1}{5! \cdot 0^5} - \cdots + (-1)^n \frac{1}{(2n+1)! \cdot 0^{2n+1}} + \cdots \\ &= \frac{1}{0} - \frac{1}{0^3} + \frac{1}{0^5} - \cdots + (-1)^n \frac{1}{0^{2n+1}} + \cdots \\ &= 0 - 0 + 0 - \cdots + (-1)^n 0 + \cdots \\ &= 0 \end{aligned}$$

$$\therefore \sin \frac{1}{0} = 0 \quad (3)$$

即ち,

$$\sin \frac{1}{0} = \sin 0 = 0 \quad (4)$$

を得る.

## 2. 余弦関数

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \quad (5)$$

ここで,  $z=1/w$  とすると,

$$\cos \frac{1}{w} = 1 - \frac{1}{2!w^2} + \frac{1}{4!w^4} - \cdots + (-1)^n \frac{1}{(2n)!w^{2n}} + \cdots \quad (6)$$

であるから,  $w=0$  を代入すると,

$$\cos \frac{1}{0} = 1 - \frac{1}{2! \cdot 0^2} + \frac{1}{4! \cdot 0^4} - \cdots + (-1)^{2n-1} \frac{1}{(2n)! \cdot 0^{2n}} + \cdots$$

$$\begin{aligned}
&= 1 - \frac{1}{0^2} + \frac{1}{0^4} - \dots + (-1)^n \frac{1}{0^{2n}} + \dots \\
&= 1 - 0 + 0 - \dots + (-1)^n 0 + \dots \\
&= 1 \\
\therefore \cos \frac{1}{0} &= 1 \tag{7}
\end{aligned}$$

即ち,

$$\cos \frac{1}{0} = \cos 0 = 1 \tag{8}$$

を得る.

### 3. 指数関数

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} + \dots \quad (-\infty < z < \infty) \tag{9}$$

ここで,  $z=1/w$  とすると,

$$e^{\frac{1}{w}} = 1 + \frac{1}{1!w} + \frac{1}{2!w^2} + \frac{1}{3!w^3} + \dots + \frac{1}{n!w^n} + \dots \quad \left(-\infty < \frac{1}{w} < \infty\right) \tag{10}$$

であるから,  $w=0$  を代入すると,

$$\begin{aligned}
e^{\frac{1}{0}} &= 1 + \frac{1}{1! \cdot 0} + \frac{1}{2! \cdot 0^2} + \frac{1}{3! \cdot 0^3} + \dots + \frac{1}{n! \cdot 0^n} + \dots \\
&= 1 + \frac{1}{0} + \frac{1}{0^2} + \frac{1}{0^3} + \dots + \frac{1}{0^n} + \dots \\
&= 1 + 0 + 0 + 0 + \dots \\
&= 1
\end{aligned}$$

$$\therefore e^{\frac{1}{0}} = 1 \tag{11}$$

即ち,

$$e^{\frac{1}{0}} = e^0 = 1 \tag{12}$$

を得る.

### 4. 対数関数

$$\log_e(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots + (-1)^{n-1} \frac{z^n}{n} + \dots \quad (-1 < z < 1) \tag{13}$$

ここで,  $z=1/w$  とすると,

$$\log_e \left(1 + \frac{1}{w}\right) = \frac{1}{w} - \frac{1}{2w^2} + \frac{1}{3w^3} - \dots + (-1)^{n-1} \frac{1}{nw^n} + \dots \quad \left(-1 < \frac{1}{w} < 1\right) \quad (14)$$

であるから,  $w=0$  を代入すると,

$$\begin{aligned} \log_e \left(1 + \frac{1}{0}\right) &= \frac{1}{0} - \frac{1}{2 \cdot 0^2} + \frac{1}{3 \cdot 0^3} - \dots + (-1)^{n-1} \frac{1}{n \cdot 0^n} + \dots \quad \left(-1 < \frac{1}{w} < 1\right) \\ &= \frac{1}{0} - \frac{1}{0^2} + \frac{1}{0^3} - \dots + (-1)^{n-1} \frac{1}{0^n} + \dots \\ &= 0 - 0 + 0 - \dots + (-1)^{n-1} 0 + \dots \\ &= 0 \end{aligned}$$

$$\therefore \log_e \left(1 + \frac{1}{0}\right) = 0 \quad (15)$$

即ち,

$$\log_e \left(1 + \frac{1}{0}\right) = \log_e(1 + 0) = \log_e(1) = 0 \quad (16)$$

を得る.

## 5. 二項関数

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{2!} z^2 + \dots + \frac{m(m-1) \cdot \dots \cdot (m-n+1)}{n!} z^n + \dots \quad (17)$$

ここで,  $z=1/w$  とすると,

$$\left(1 + \frac{1}{w}\right)^m = 1 + \frac{m}{w} + \frac{m(m-1)}{2! w^2} + \dots + \frac{m(m-1) \cdot \dots \cdot (m-n+1)}{n! w^n} + \dots \quad (18)$$

であるから,  $w=0$  を代入すると,

$$\begin{aligned} \left(1 + \frac{1}{0}\right)^m &= 1 + \frac{m}{0} + \frac{m(m-1)}{2! \cdot 0^2} + \dots + \frac{m(m-1) \cdot \dots \cdot (m-n+1)}{n! \cdot 0^n} + \dots \\ &= 1 + 0 + 0 + \dots \\ &= 1 \end{aligned}$$

$$\therefore \left(1 + \frac{1}{0}\right)^m = 1 \quad (19)$$

即ち,

$$\left(1 + \frac{1}{0}\right)^m = (1+0)^m = 1^m = 1 \quad (20)$$

を得る.