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Series expansion of elementary functions and division by zero

1. The sine function can be expressed as

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots \qquad (1)$$

Here, if we let $z = 1/w$, then

$$\sin\frac{1}{w} = \frac{1}{w} - \frac{1}{3!\,w^3} + \frac{1}{5!\,w^5} - \cdots + (-1)^n \frac{1}{(2n+1)!\,w^{2n+1}} + \cdots \qquad (2)$$

Therefore, substituting $w = 0$,

$$\begin{aligned}\sin\frac{1}{0} &= \frac{1}{0} - \frac{1}{3!\cdot 0^3} + \frac{1}{5!\cdot 0^5} - \cdots + (-1)^n \frac{1}{(2n+1)!\cdot 0^{2n+1}} + \cdots \\ &= \frac{1}{0} - \frac{1}{0^3} + \frac{1}{0^5} - \cdots + (-1)^n \frac{1}{0^{2n+1}} + \cdots \\ &= 0 - 0 + 0 - \cdots + (-1)^n 0 + \cdots \\ &= 0\end{aligned}$$

$$\therefore\ \sin\frac{1}{0} = 0 \qquad (3)$$

In other words, we obtain

$$\sin\frac{1}{0} = \sin 0 = 0 \qquad (4)$$

2. Cosine function

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \qquad (5)$$

Here, if we let $z = 1/w$, then

$$\cos\frac{1}{w} = 1 - \frac{1}{2!\,w^2} + \frac{1}{4!\,w^4} - \cdots + (-1)^n \frac{1}{(2n)!\,w^{2n}} + \cdots \qquad (6)$$

Therefore, substituting $w = 0$,

$$\cos\frac{1}{0} = 1 - \frac{1}{2!\cdot 0^2} + \frac{1}{4!\cdot 0^4} - \cdots + (-1)^{2n-1} \frac{1}{(2n)!\cdot 0^{2n}} + \cdots$$

$$= 1 - \frac{1}{0^2} + \frac{1}{0^4} - \cdots + (-1)^n \frac{1}{0^{2n}} + \cdots$$

$$= 1 - 0 + 0 - \cdots + (-1)^n 0 + \cdots$$

$$= 1$$

$$\therefore \cos\frac{1}{0} = 1 \quad (7)$$

In other words, we obtain

$$\cos\frac{1}{0} = \cos 0 = 1 \quad (8)$$

3. Exponential function

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \quad (-\infty < z < \infty) \quad (9)$$

Here, if we let $z = 1/w$, then

$$e^{\frac{1}{w}} = 1 + \frac{1}{1!\,w} + \frac{1}{2!\,w^2} + \frac{1}{3!\,w^3} + \cdots + \frac{1}{n!\,w^n} + \cdots \quad \left(-\infty < \frac{1}{w} < \infty\right) \quad (10)$$

Therefore, substituting $w = 0$,

$$e^{\frac{1}{0}} = 1 + \frac{1}{1! \cdot 0} + \frac{1}{2! \cdot 0^2} + \frac{1}{3! \cdot 0^3} + \cdots + \frac{1}{n! \cdot 0^n} + \cdots$$

$$= 1 + \frac{1}{0} + \frac{1}{0^2} + \frac{1}{0^3} + \cdots + \frac{1}{0^n} + \cdots$$

$$= 1 + 0 + 0 + 0 + \cdots$$

$$= 1$$

$$\therefore e^{\frac{1}{0}} = 1 \quad (11)$$

In other words, we obtain

$$e^{\frac{1}{0}} = e^0 = 1 \quad (12)$$

4. Logarithm function

$$\log_e(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots + (-1)^{n-1}\frac{z^n}{n} + \cdots \quad (-1 < z < 1) \quad (13)$$

Here, if we let $z = 1/w$, then

$$\log_e\left(1 + \frac{1}{w}\right) = \frac{1}{w} - \frac{1}{2w^2} + \frac{1}{3w^3} - \cdots + (-1)^{n-1}\frac{1}{nw^n} + \cdots \quad \left(-1 < \frac{1}{w} < 1\right) \quad (14)$$

Therefore, substituting $w = 0$,

$$\log_e\left(1+\frac{1}{0}\right)=\frac{1}{0}-\frac{1}{2\cdot 0^2}+\frac{1}{3\cdot 0^3}-\cdots+(-1)^{n-1}\frac{1}{n\cdot 0^n}+\cdots \quad \left(-1<\frac{1}{w}<1\right)$$

$$=\frac{1}{0}-\frac{1}{0^2}+\frac{1}{0^3}-\cdots+(-1)^{n-1}\frac{1}{0^n}+\cdots$$

$$=0-0+0-\cdots+(-1)^{n-1}0+\cdots$$

$$=0$$

$$\therefore \quad \log_e\left(1+\frac{1}{0}\right)=0 \tag{15}$$

In other words, we obtain

$$\log_e\left(1+\frac{1}{0}\right)=\log_e(1+0)=\log_e(1)=0 \tag{16}$$

5. A binary function

$$(1+z)^m=1+mz+\frac{m(m-1)}{2!}z^2+\cdots+\frac{m(m-1)\cdot\cdots\cdot(m-n+1)}{n!}z^n+\cdots \tag{17}$$

Here, if we let $z = 1/w$, then

$$\left(1+\frac{1}{w}\right)^m=1+\frac{m}{w}+\frac{m(m-1)}{2!\,w^2}+\cdots+\frac{m(m-1)\cdot\cdots\cdot(m-n+1)}{n!\,w^n}+\cdots \tag{18}$$

Therefore, substituting $w = 0$,

$$\left(1+\frac{1}{0}\right)^m=1+\frac{m}{0}+\frac{m(m-1)}{2!\cdot 0^2}+\cdots+\frac{m(m-1)\cdot\cdots\cdot(m-n+1)}{n!\cdot 0^n}+\cdots$$

$$=1+0+0+\cdots$$

$$=1$$

$$\therefore \quad \left(1+\frac{1}{0}\right)^m=1 \tag{19}$$

In other words, we obtain

$$\left(1+\frac{1}{0}\right)^m=(1+0)^m=1^m=1 \tag{20}$$