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Functional remainder formula division by zero and zero to the power of zero

Theorem: For a dividend function *P*(*x*), divisor function Q(*x*), quotient function *A*(*x*), and remainder function *R*(*x*) that satisfy the following relationship,

$$P(x) = Q(x)A(x) + R(x)$$

the following holds:

$$\frac{P(x)}{Q(x)} = Q^0(x)A(x) \cdots R(x)$$

In particular, for a value *a* that satisfies Q(*a*) = 0, the following holds:

$$\frac{R(a)}{0} = 0 \cdots R(a)$$

Note that $Q^0(x)$ refers to Q^0, and

$$Q^0(x) = \begin{cases} 0 & (Q = 0) \\ 1 & (Q \neq 0) \end{cases}$$

Proof: Based on the assumptions, for a dividend function *P*(*x*), divisor function Q(*x*), quotient function *A*(*x*), and remainder function *R*(*x*),

$$P(x) = Q(x)A(x) + R(x) \qquad (1)$$

Therefore, dividing both sides by Q(*x*), this can be expressed as

$$\frac{P(x)}{Q(x)} = \frac{Q(x)}{Q(x)}A(x) \cdots R(x) \qquad (2)$$

Note that 0/0 = 0, and reduction is possible only when Q/Q = 1 holds. Therefore, we choose not to reduce Equation (2).

The Q/Q part in the right-hand side of Equation (2) is

$$\frac{Q(x)}{Q(x)} = Q^{+1}(x) \cdot Q^{-1}(x) = Q^{1-1}(x) = Q^0(x) \qquad (3)$$

For a value of *a* that satisfies Q(*a*) = 0, we obtain

$$Q^0(a) = \frac{Q(a)}{Q(a)} = \frac{0}{0} = 0 \qquad (4)$$

Conversely, for a value of *b* that satisfies Q(*b*) ≠ 0, we obtain

$$Q^0(b) = \frac{Q(b)}{Q(b)} = 1 \qquad (5)$$

Therefore, based on Equation (4) and Equation (5),

$$Q^0(x) = \begin{cases} 0 & (Q = 0) \\ 1 & (Q \neq 0) \end{cases} \qquad (6)$$

holds. Applying Equation (3) and Equation (6) to Equation (2), we obtain

$$\frac{P(x)}{Q(x)} = Q^0(x)A(x) \cdots R(x) \quad Q^0(x) = \begin{cases} 0 & (Q = 0) \\ 1 & (Q \neq 0) \end{cases} \qquad (7)$$

Based on Equation (1),

$$Q(a) = 0 \Rightarrow R(a) = P(a) \qquad (8)$$

Therefore, if we let $x = a$ in Equation (7), then the left-hand side is

$$\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} = \frac{P(a)}{0} = \frac{R(a)}{0} \qquad (9)$$

and the right-hand side is

$$Q^0(x)A(x) = Q^0(a)A(a) = 0 \cdot A(a) = 0 \cdots R(a) \qquad (10)$$

Therefore, based on Equation (9) and (10), we obtain

$$\frac{R(a)}{0} = 0 \cdots R(a) \qquad (11)$$

□

It is necessary to be aware that $A(a) = 0$ does not hold in general.

Consider a specific example of functional remainder formula division by zero.

If we let the dividend function *P*(*x*) be

$$P(x) = 2x^2 - 3x + 1 \qquad (12)$$

and the divisor function Q(x) be

$$Q(x) = x + 1 \qquad (13)$$

then the quotient function $A(x)$ is

$$A(x) = 2x - 5 \qquad (14)$$

and the remainder function $R(x)$ is

$$R(x) = 6 \qquad (15)$$

Therefore, substituting Equation (12), (13), (14), and (15) into Equation (7), this becomes

$$\frac{2x^2 - 3x + 1}{x + 1} = (x + 1)^0(2x - 5) \cdots 6 \qquad (16)$$

If we define $f(x)$ as the left-hand side of Equation (16), and g(x) as the right-hand side, then when $x = b \neq -1$,

$$Q(b) \neq 0 \qquad (17)$$

Therefore, Equation (16) is

$$\frac{2b^2 - 3b + 1}{b + 1} = (2b - 5) \cdots 6 \qquad (18)$$

and

$$2b^2 - 3b + 1 = (b + 1)(2b - 5) + 6 \qquad (19)$$

holds and satisfies the relationship in Equation (1).

When $x = -1$,

$$Q(-1) = 0 \qquad (20)$$

Therefore, $f(x)$ in Equation (16) is

$$f(-1) = \frac{2(-1)^2 - 3(-1) + 1}{(-1) + 1} = \frac{6}{0} \qquad (21)$$

and $g(x)$ is

$$g(-1) = \{(-1) + 1\}^0\{2 \cdot (-1) - 5\} = 0^0 \cdot (-7) = 0 \cdot (-7) = 0 \cdots 6 \qquad (22)$$

The transformation in Equation (22) uses the fundamental theorem of zero to the power of zero, $0^0 = 0$. Therefore, based on Equation (21) and Equation (22), we obtain

$$\frac{6}{0} = 0 \cdots 6 \qquad (23)$$

In other words, this implies

$$6 = 0 \cdot 0 + 6 \qquad (24)$$

Note that in this example,

$$A(-1) = -7 \qquad (25)$$

It should be noted that $A(a) = 0$ does not necessarily hold.