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The general inverse and division by zero

Let us consider an attempt to generalize the inverse by extending the inverse in a natural form and including the concept of division by zero.

Definition: The inverse $(\alpha/\beta)^{-1}$ of a fraction α/β involving two real numbers α and β is defined as

$$\left(\frac{\alpha}{\beta}\right)^{-1} = \left(\frac{\alpha}{\beta}\right)^{0} \frac{\beta}{\alpha}$$

Note that

$$\left(\frac{\alpha}{\beta}\right)^{0} = \begin{cases} 0 & (\alpha = 0 \vee \beta = 0) \\ 1 & (\alpha \neq 0 \wedge \beta \neq 0) \end{cases}$$

According to the definition, if we let

$$\frac{\alpha}{\beta} = \gamma \qquad (1)$$

for a certain number α and a certain number β, then the inverse is

$$\left(\frac{\alpha}{\beta}\right)^{-1} = \gamma^{-1} \qquad (2)$$

Therefore, in the case $\alpha = \beta$, Equation (1) is

$$\frac{\alpha}{\alpha} = \gamma \qquad (3)$$

and Equation (2) is

$$\left(\frac{\alpha}{\alpha}\right)^{-1} = \gamma^{-1} \qquad (4)$$

From the definition, we obtain

$$\gamma = \frac{\alpha}{\alpha} = \left(\frac{\alpha}{\alpha}\right)^{-1} = \gamma^{-1} \qquad (5)$$

Therefore,

$$\gamma = \gamma^{-1} \quad (\alpha = \beta) \qquad (6)$$

holds. The value of γ can take on one of two values depending on the value of $\alpha = \beta$, and becomes

$$0 = 0^{-1} \quad (\alpha = \beta = 0) \qquad (7)$$

$$1 = 1^{-1} \quad (\alpha \neq 0 \land \alpha = \beta) \qquad (8)$$

Equation (7) implies that the inverse of 0, or in other words 0^{-1}, is not 1/0, and is limited to 0. In addition, if we consider that $\alpha/\beta = (\alpha/\beta)^1$ and $\gamma = (\gamma)^1$, taking the product of each side of Equation (1) and Equation (2), we obtain

$$\left(\frac{\alpha}{\beta}\right)^1 \times \left(\frac{\alpha}{\beta}\right)^{-1} = \gamma^1 \times \gamma^{-1} \qquad (9)$$

Therefore, taking the exponent of the left-hand side of Equation (9), this becomes

$$\left(\frac{\alpha}{\beta}\right)^1 \times \left(\frac{\alpha}{\beta}\right)^{-1} = \left(\frac{\alpha}{\beta}\right)^{1-1} = \left(\frac{\alpha}{\beta}\right)^0 \qquad (10)$$

Therefore, we obtain

$$\frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = \left(\frac{\alpha}{\beta}\right)^0 \qquad (11)$$

and

$$\left(\frac{\alpha}{\beta}\right)^{-1} = \left(\frac{\alpha}{\beta}\right)^0 \frac{\beta}{\alpha} \qquad (12)$$

Equations (11) and (12) are a stricter definition of the inverse and can consist of a natural extension to the concept of the inverse. In other words, Equation (11) and Equation (12) are the equations that define the general inverse.

Therefore, the inverse of α/β for a given number α and a given number β, or in other words, $(\alpha/\beta)^{-1}$, satisfies Equation (12). Therefore, when at least one of α or β are 0, based on the fundamental theorem of division by zero $a/0 = 0$ and the fundamental theorem of the power of zero $0^0 = 0$, we obtain

$$\left(\frac{\alpha}{\beta}\right)^0 = \left(\frac{0}{\beta}\right)^0 = 0^0 = 0 \qquad (13)$$

$$\left(\frac{\alpha}{\beta}\right)^0 = \left(\frac{\alpha}{0}\right)^0 = 0^0 = 0 \qquad (14)$$

$$\left(\frac{\alpha}{\beta}\right)^0 = \left(\frac{0}{0}\right)^0 = 0^0 = 0 \qquad (15)$$

These equations show that the inverse is limited to 0. Note that the relationship in Equation (14) can be algebraically expressed as

$$\frac{a}{0} = \frac{a}{0}\cdot 1 = \frac{a}{0}\cdot\frac{b}{b} = \frac{a\times b}{0\times b} = \frac{b\times a}{0} = b\cdot\frac{a}{0} \quad (b \neq 0{,}1) \qquad (16)$$

when $a \neq 0$. Based on this, we obtain

$$\frac{a}{0} = \frac{0}{b-1} = 0 \qquad (17)$$

Therefore,

$$\frac{a}{0} = 0 \qquad (18)$$

holds (It is necessary to be aware of the difference between fractions and division). Therefore, when at least one of α or β are 0, the inverse is

$$\left(\frac{0}{\beta}\right)^{-1} = \left(\frac{0}{\beta}\right)^0 \frac{\beta}{0} = 0\cdot 0 = 0 \qquad (19)$$

$$\left(\frac{\alpha}{0}\right)^{-1} = \left(\frac{\alpha}{0}\right)^0 \frac{0}{\alpha} = 0\cdot 0 = 0 \qquad (20)$$

$$\left(\frac{0}{0}\right)^{-1} = \left(\frac{0}{0}\right)^0 \frac{0}{0} = 0\cdot 0 = 0 \qquad (21)$$

Based on these results, it can be said that $0^{-1} = 0$ uniquely holds. This result agrees with the result that Equations (19), (20), and (21) become

$$0^{-1} = \left(\left(\frac{0}{\beta}\right)^{-1}\right)^{-1} = \left(\frac{0}{\beta}\right)^{+1} = 0^1 = 0 \qquad (22)$$

$$0^{-1} = \left(\left(\frac{\alpha}{0}\right)^{-1}\right)^{-1} = \left(\frac{\alpha}{0}\right)^{+1} = 0^1 = 0 \qquad (23)$$

$$0^{-1} = \left(\left(\frac{0}{0}\right)^{-1}\right)^{-1} = \left(\frac{0}{0}\right)^{+1} = 0^1 = 0 \qquad (24)$$

respectively when the inverse of both sides are taken with the right-most side and the left-most side switched. Therefore, the relationship in Equation (7)

$$0^{-1} = 0$$

holds uniquely.

In contrast, when neither α nor β are 0,

$$\left(\frac{\alpha}{\beta}\right)^0 = \gamma^0 = 1 \qquad (25)$$

holds. Therefore, when neither α nor β are 0, the inverse of α/β, or in other words $(\alpha/\beta)^{-1}$, is given by

$$\left(\frac{\alpha}{\beta}\right)^{-1} = \left(\frac{\alpha}{\beta}\right)^0 \frac{\beta}{\alpha} = 1 \cdot \frac{\beta}{\alpha} = \frac{\beta}{\alpha} \qquad (26)$$

similar to the conventional inverse. Based on this, when at least one of α or β are 1, this becomes

$$\left(\frac{1}{\beta}\right)^{-1} = \left(\frac{1}{\beta}\right)^0 \frac{\beta}{1} = 1 \cdot \beta = \beta \qquad (27)$$

$$\left(\frac{\alpha}{1}\right)^{-1} = \left(\frac{\alpha}{1}\right)^0 \frac{1}{\alpha} = 1 \cdot \frac{1}{\alpha} = \frac{1}{\alpha} \qquad (28)$$

$$\left(\frac{1}{1}\right)^{-1} = \left(\frac{1}{1}\right)^0 \frac{1}{1} = 1 \cdot 1 = 1 \qquad (29)$$

In other words, based on Equation (7) and (28), we obtain

$$\alpha^{-1} = \begin{cases} 0 & (\alpha = 0) \\ \dfrac{1}{\alpha} & (\alpha \neq 0) \end{cases} \qquad (30)$$

Based on this, the operation

$$\times\, \alpha^{-1} = \begin{cases} \times\, 0 & (\alpha = 0) \\ \times \dfrac{1}{\alpha} & (\alpha \neq 0) \end{cases} \qquad (31)$$

holds. Conversely,

$$\div\, \alpha = \begin{cases} \times\, 0^{-1} = \times\, 0 & (\alpha = 0) \\ \times \dfrac{1}{\alpha} & (\alpha \neq 0) \end{cases} \qquad (32)$$

holds. Therefore, based on Equation (31) and Equation (32), the operation replacement relationship

$$\times\, \alpha^{-1} \Leftrightarrow \div\, \alpha \qquad (33)$$

holds. This shows that it is possible to replace the reverse exponent and division (it is necessary to use caution regarding the fact that this is limited to cases in which the remainder form is not employed, and only the quotient is considered).