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Hiroshi Michiwaki

**Orthogonality conditions for two straight lines, the structure of the Cartesian coordinates, and division by zero**

Two straight lines $y = ax + b$ and $y = a'x + b'$ are said to be orthogonal to each other if the following condition holds:

If the slopes of the lines are

$$a = \tan\alpha\,,\quad a' = \tan\alpha' \qquad (1)$$

respectively, then the lines are orthogonal if

$$\alpha' - \alpha = \pm\frac{\pi}{2} \qquad \therefore \quad \alpha' = \alpha \pm \frac{\pi}{2} \qquad (2)$$

Based on this, Equation (1) can be expressed as

$$a' = \tan\alpha' = \tan\left(\alpha \pm \frac{\pi}{2}\right) = -\cot\alpha = -(\tan\alpha)^{-1} \qquad (3)$$

Therefore,

$$a \cdot a' = \tan\alpha \cdot \tan\alpha' = \tan\alpha \cdot (-\cot\alpha) = -\tan\alpha \cdot (\tan\alpha)^{-1} \qquad (4)$$

holds. Equation (4) can be transformed further to obtain

$$-\tan\alpha \cdot (\tan\alpha)^{-1} = -\left(\frac{\sin\alpha}{\cos\alpha}\right) \cdot \left(\frac{\sin\alpha}{\cos\alpha}\right)^{-1} = -\left(\frac{\sin\alpha}{\cos\alpha}\right) \cdot \left(\frac{\cos\alpha}{\sin\alpha}\right) \qquad (5)$$

When the included angle $\alpha$ is

$$\alpha \neq \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\} \qquad (6)$$

Equation (5) is clearly $-1$, so

$$a \cdot a' = -\tan\alpha \cdot (\tan\alpha)^{-1} = -\left(\frac{\sin\alpha}{\cos\alpha}\right) \cdot \left(\frac{\sin\alpha}{\cos\alpha}\right)^{-1} = -1 \qquad (7)$$

On the other hand, when the included angle $\alpha$ in Equation (5) is

$$\alpha = \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\} \qquad (8)$$

if we let $k$ be an integer, $\alpha$ can be separated into the cases

$$\alpha = k\pi \qquad (9)$$

and

$$\alpha = \frac{2k-1}{2}\pi \qquad (10)$$

In the case of Equation (9),

$$-\tan k\pi \cdot (\tan k\pi)^{-1} = -\left(\frac{\sin k\pi}{\cos k\pi}\right) \cdot \left(\frac{\sin k\pi}{\cos k\pi}\right)^{-1} = -\left(\frac{0}{\pm 1}\right) \cdot \left(\frac{0}{\pm 1}\right)^{-1} = -\frac{0}{1} \cdot \frac{1}{0} = -\frac{0}{0} = 0 \quad (11)$$

and in the case of Equation (10),

$$-\tan\left(\frac{2k-1}{2}\pi\right) \cdot \left\{\tan\left(\frac{2k-1}{2}\pi\right)\right\}^{-1} = -\left\{\frac{\sin\left(\frac{2k-1}{2}\pi\right)}{\cos\left(\frac{2k-1}{2}\pi\right)}\right\} \cdot \left\{\frac{\sin\left(\frac{2k-1}{2}\pi\right)}{\cos\left(\frac{2k-1}{2}\pi\right)}\right\}^{-1}$$

$$= -\left(\frac{\pm 1}{0}\right) \cdot \left(\frac{\pm 1}{0}\right)^{-1} = -\frac{1}{0} \cdot \frac{0}{1} = -\frac{0}{0} = 0 \quad (12)$$

Note that 0/0 = 0 follows from the fundamental theorem of division by zero. Therefore, if the included angle $\alpha$ satisfies Equation (8), then

$$a \cdot a' = -\tan \alpha \cdot (\tan \alpha)^{-1} = -\left(\frac{\sin \alpha}{\cos \alpha}\right) \cdot \left(\frac{\sin \alpha}{\cos \alpha}\right)^{-1} = 0 \qquad (13)$$

holds. In other words, by combining Equation (7) and Equation (13), we obtain

$$a \cdot a' = -\tan \alpha \cdot (\tan \alpha)^{-1} = -\left(\frac{\sin \alpha}{\cos \alpha}\right) \cdot \left(\frac{\sin \alpha}{\cos \alpha}\right)^{-1} = \begin{cases} 0 & \alpha = \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\} \\ -1 & \alpha \neq \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\} \end{cases} \quad (14)$$

Conversely, based on Equation (7) and Equation (13),

$$-\left(\frac{\sin \alpha}{\cos \alpha}\right) \cdot \left(\frac{\sin \alpha}{\cos \alpha}\right)^{-1} = -\left(\frac{\sin \alpha}{\cos \alpha}\right)^{1-1} = -\left(\frac{\sin \alpha}{\cos \alpha}\right)^{0} = -\tan^0 \alpha$$

$$\therefore \quad -\left(\frac{\sin \alpha}{\cos \alpha}\right)^{0} = -\tan^0 \alpha = \begin{cases} 0 & \alpha = \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\} \\ -1 & \alpha \neq \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\} \end{cases} \qquad (15)$$

holds. In particular, if we let $\alpha = \pi/2$, then we obtain the relationship

$$-\left(\frac{\sin \pi/2}{\cos \pi/2}\right) \cdot \left(\frac{\cos \pi/2}{\sin \pi/2}\right) = -\frac{1}{0} \cdot \frac{0}{1} = -\frac{1 \cdot 0}{0 \cdot 1} = -\frac{0}{0} = -0$$

$$= -\tan^0 \pi/2 = -0^0$$

$$\therefore \quad 0^0 = \frac{0}{0} = 0 \qquad (16)$$

Incidentally, if we let $\theta$ represent the angle formed by the two straight lines $y = ax + b$ and $y = a'x + b'$, this can be expressed as

$$\tan\theta = \tan(\alpha' - \alpha) = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha' \cdot \tan\alpha} \qquad (17)$$

The two straight lines are orthogonal, so $\theta = \pi/2$. Therefore, when the included angle $\alpha$ is

$$\alpha \neq \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\}$$

based on Equation (4) and Equation (7),

$$\tan\frac{\pi}{2} = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha' \cdot \tan\alpha} = -\frac{\cot\alpha + \tan\alpha}{1 - 1} = \frac{\frac{\cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{\cos\alpha}}{0} = \frac{\frac{\cos^2\alpha + \sin^2\alpha}{\sin\alpha \cdot \cos\alpha}}{0} = \frac{1}{0} = 0 \quad (18)$$

and when the included angle $\alpha$ is

$$\alpha = \left\{k\pi \vee \frac{2k-1}{2}\pi \,\middle|\, k \in \mathrm{Z}\right\}$$

based on Equation (4) and Equation (14), we obtain

$$\tan\frac{\pi}{2} = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha' \cdot \tan\alpha} = -\frac{\cot\alpha + \tan\alpha}{1 + 0} = \frac{\frac{\cos^2\alpha + \sin^2\alpha}{\sin\alpha \cdot \cos\alpha}}{1} = \frac{\frac{1}{0}}{1} = \frac{0}{1} = 0 \quad (19)$$

Note that 1/0 = 0 follows from division by zero.

Based on the above result, it can be said that the slope between two straight lines that are orthogonal to each other is 0, regardless of the Cartesian coordinate system. Interestingly, in the situation where the two straight lines are parallel to each other, or in other words when $\theta = 0$ and $\alpha = \alpha'$, Equation (17) becomes

$$\tan\theta = \tan 0 = \frac{\tan\alpha - \tan\alpha}{1 + \tan^2\alpha} = \frac{0}{1 + \tan^2\alpha} \qquad (20)$$

and when $\alpha \neq \pi/2$, Equation (20) clearly becomes 0. On the other hand, when $\alpha = \pi/2$, Equation (20) becomes

$$\tan 0 = \frac{0}{1 + \tan^2\pi/2} = \frac{0}{1 + 0^2} = \frac{0}{1} = 0 \qquad (21)$$

As this is equal to 0, the slope with respect to all included angles $\alpha$ in two straight lines that are parallel to each other is 0, and is equal to the slope of two straight lines that are parallel to each other.

The above result shows that when a straight line L' is tilted with included angle of $\theta$ with respect to another straight line L, if $\theta = 0$, then the slope $a$ between the two straight lines is 0, and $\theta = \pi/2 \Rightarrow a = 0$, $\theta = \pi \Rightarrow a = 0$, $\theta = 3\pi/2 \Rightarrow a = 0$, and $\theta = 2\pi \Rightarrow a = 0$.

In other words, an orthogonal coordinate system is composed of units of half-open angular intervals every $\pi/2$ (every quadrant) from the coordinate axis that serves as a reference, such as $0 \leqq \theta < \pi/2$ (the first quadrant), $\pi/2 \leqq \theta < \pi$ (the second quadrant), $\pi \leqq \theta < 3\pi/2$ (the third quadrant), and $3\pi/2 \leqq \theta < 2\pi$ (the fourth quadrant). The borders between neighboring quadrants are connected with discontinuities that are strong for the slope of the coordinate axis.

In other words, using the first quadrant as an example, $0 \leqq \theta < \pi/2 \Rightarrow 0 \leqq a < \infty$, and in particular, $\theta \rightarrow \pi/2 \Rightarrow a \rightarrow \infty$, and $\theta = \pi/2 \Rightarrow a = 0$. The last relationship shows that the target range has transitioned to the range of the next unit (the second quadrant) as viewed from the previous unit (the first quadrant).

Below, we decompose the four quadrants in the *x*-*y* Cartesian coordinate system, from the first quadrant to the fourth quadrant, into separate quadrants (hereafter referred to as quadrant decomposition), and show the structure of the Cartesian coordinate system in Fig. 1. As shown in Fig. 1, for the range of slopes of the axes, or in other words, the ranges of the axes' included angles, the side with the larger included angle is an open angle (a concept that corresponds to the concept of an open point) in contrast to the side with the smaller included angle, which is a closed angle (a concept that corresponds to the concept of a closed point). The range of the included angle of each quadrant is a half-open angle.

Next, we synthesize the *x*-*y* Cartesian coordinates that were decomposed into quadrants into the original *x*-*y* Cartesian coordinates formed from the first quadrant through the fourth quadrant. The relationship of the axis slope that corresponds to the minimum value and the maximum value in the included angle range for each quadrant within the region of each quadrant is shown in Fig. 2. The symbol $\breve{0}$ written in Fig. 2 refers to an infinitesimally small number that is smaller than 0 and that is larger than any negative constant.

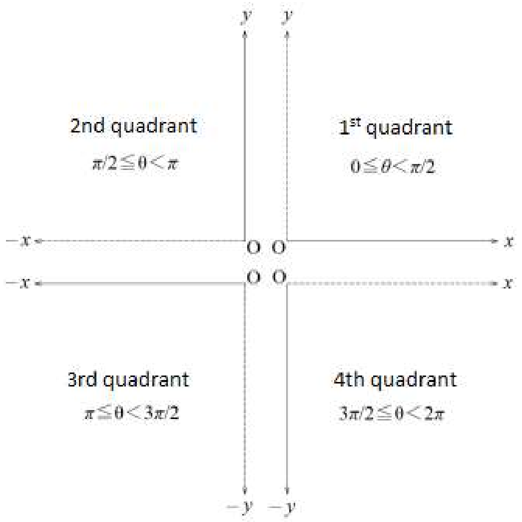


Fig. 1: Structure of the *x*-*y* Cartesian coordinates decomposed into 4 quadrants and their ranges

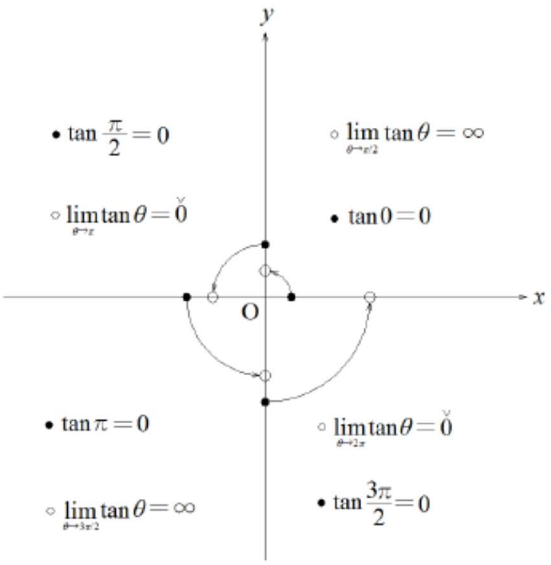


Fig. 2: Relationship between the *x*-*y* Cartesian coordinates and the axis slope due to 4 quadrant synthesis