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General Trigonometric Functions and Division by Zero

Even though it would typically be proper to follow the process of definition, theorem, and proof, the definitions are explained in detail in the proof below. Consequently, the assertions of the following theorems make use of the definitions in the proof.

Theorem 1 For the extended general cosine function $\cos_r \theta$, where θ is the radial angle, with respect to the x-axis, of a certain radius vector r extending from the origin O on an xy-coordinate plane, the following holds true:

$$\cos_r 0 = \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases}$$

Proof For a certain radius r extending from the origin O on an xy-coordinate plane, if θ is its angle with respect to the x-axis, its cosine is expressed as

$$\cos \theta_r = \frac{x_r}{r} \qquad (1)$$

If now $r \neq 0$ and θ is defined as $0 \leqq \theta_{r \neq 0} = \pi/2 \vee 3\pi/2 < 2\pi$, then, $x_{r \neq 0} \neq 0$. Thus,

$$\cos \theta_{r \neq 0} = 0 \qquad (2)$$

Furthermore, if θ is defined as $0 \leqq \theta_{r \neq 0} = 0 < 2\pi$,

$$\cos \theta_{r \neq 0} = \cos 0 = 1 \qquad (3)$$

However, as $r = 0 \Rightarrow x_{r=0} = 0 \wedge \theta_{r=0} = 0$; if $r = 0$, then the left-hand side of (1) becomes

$$\cos \theta_r = \cos \theta_0 = \cos 0 \qquad (4)$$

and the right-hand side becomes

$$\frac{x_r}{r} = \frac{x_0}{0} = \frac{0}{0} = 0 \qquad (5)$$

This result depends on the fundamental theorem of division by zero, 0/0 = 0. Therefore, (4) and (5) imply that

$$\cos \theta_{r=0} = \cos 0 = 0 \qquad (6)$$

That is, by (3) and (6), we have

$$\cos 0 = \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases} \qquad (7)$$

Equations (4), (5), and (6) assert that

$$[\![\cos\theta_r]\!]_{r=0} = \left[\!\left[\frac{x_r}{r}\right]\!\right]_{r=0} = 0 \qquad (8)$$

and (8) gives the impression that the value of the zero cosine function (cos0) could be either 0 or 1. In practice, in addition to the value of θ, whether the function takes the value of 0 or 1 depends on the radius r, which is linearly independent of θ. Thus, it is not considered a single-variable function. In addition, as θ is sometimes dependent on r, the notation θ_r is considered inappropriate. Therefore, by letting

$$\theta_{r=0} = \theta_0 = 0 \qquad \text{(a)}$$

and considering only the result of (a), i.e., 0 on the rightmost side, it is impossible to determine if it was obtained when $r = 0$ or $r \neq 0$. Appending a subscript to the result to leave a trace of r and make it possible to determine how the result was derived then yields

$$\theta_{r=0} = \theta_0 = 0_0 \qquad \text{(b)}$$

If the notation in (b) is used, there is a certain rationality to the fact that it can be determined whether $r = 0$ or $r \neq 0$ by mere inspection of the result of the equation, i.e., 0_0 on the rightmost side. This may also give the impression that there are different types of 0.

For the above reasons, although trigonometric functions have so far been treated as one-variable functions of θ, they are actually two-variable functions of r and θ. In the sense discussed here, trigonometric functions have so far been considered only in cases fulfilling the condition $r \neq 0$. Thus, trying to avoid the problem of non-uniqueness by appending a subscript to variables will arguably result in some degree of inconvenience.

Accordingly, the following is an extension of the definition of the notation of trigonometric functions.

For a radius r, the sine, cosine, and tangent of the angle θ between r and the reference line are expressed as

$$\sin_r\theta \equiv \frac{y_r}{r}, \quad \cos_r\theta \equiv \frac{x_r}{r}, \quad \tan_r\theta \equiv \frac{y_r}{x_r} = \frac{\sin_r\theta}{\cos_r\theta} \qquad (9)$$

respectively. Thus, when $r \neq 0$,

$$\sin_{r\neq 0}\theta \equiv \sin\theta\,, \quad \cos_{r\neq 0}\theta \equiv \cos\theta\,, \quad \tan_{r\neq 0}\theta \equiv \tan\theta \qquad (10)$$

and $r = 0 \Rightarrow \theta = 0$, in which case

$$\sin_0\theta = \frac{y_0}{0} = \frac{0}{0} = 0, \quad \cos_0\theta = \frac{x_0}{0} = \frac{0}{0} = 0, \quad \tan_0\theta = \frac{y_0}{x_0} = \frac{\sin_0\theta}{\cos_0\theta} = \frac{0}{0} = 0 \qquad (11)$$

However, the result of (11) depends on the fundamental theorem of division by zero, 0/0 = 0. Consequently, (10) and (11) imply that

$$\cos_{r\neq 0} 0 = \cos 0 = 1 \wedge \cos_0 0 = 0$$

$$\therefore \quad \cos_r 0 = \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases} \qquad (12)$$

when $\theta = 0$.

QED.

Theorem 2 For the extended general sine function $\sin_r \theta$, where θ is the radial angle, with respect to the x-axis, of a certain radius vector r extending from the origin O on an xy-coordinate plane, the following holds true:

$$\sin_r 0 = 0 \quad (r = 0 \vee r \neq 0)$$

Proof Using (11) and applying $r = 0 \Rightarrow \theta = 0$ to the definition of the general sine function in (9) yields

$$\sin_0 \theta = \sin_0 0 = \frac{y_0}{0} = \frac{0}{0} = 0 \qquad (13)$$

Furthermore, using (10), when $r \neq 0$, and letting $\theta = 0$ yields

$$\sin_{r\neq 0} \theta = \sin_{r\neq 0} 0 = \sin 0 = 0 \qquad (14)$$

Consequently, based on (13) and (14), the value of the sine function is 0 when θ=0, independent of the value of r. Therefore,

$$\sin_r 0 = 0 \quad (r = 0 \vee r \neq 0) \qquad (15)$$

QED.

Theorem 3 For the extended general tangent function $\tan_r \theta$, where θ is the radial angle, with respect to the x-axis, of a certain radius vector r extending from the origin O on an xy-coordinate plane, the following holds true:

$$\tan_r 0 = 0 \quad (r = 0 \vee r \neq 0)$$

Proof Using (11) and applying r=0 $\Rightarrow$ θ=0 to the definition of the general tangent function in (9) yields

$$\tan_0 \theta = \tan_0 0 = \frac{\sin_0 \theta}{\cos_0 \theta} \qquad (16)$$

and by Theorems 1 and 2, it is clear that

$$\frac{\sin_0 \theta}{\cos_0 \theta} = \frac{\sin_0 0}{\cos_0 0} = \frac{0}{0} = 0 \qquad (17)$$

Furthermore, using (10), when $r \neq 0$, and letting θ=0 yields

$$\tan_{r\neq 0} \theta = \tan_{r\neq 0} 0 = \sin 0 = 0 \qquad (18)$$

Consequently, by (17) and (18), the value of the tangent function is 0 when $\theta = 0$, independent of the value of r. Therefore,

$$\tan_r 0 = 0 \quad (r = 0 \vee r \neq 0) \qquad (19)$$

QED.