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Division by Zero and Limits

For a that satisfies $\boldsymbol{a} \in \mathrm{R} \wedge \boldsymbol{a} \neq 0$, take $\boldsymbol{n}$ as $\boldsymbol{a}$ real number of an arbitrary size, and consider dividing a with 0.

$$\frac{a}{0} = \frac{a}{0} \times 1 = \frac{a}{0} \times \frac{1/n}{1/n} = \frac{a}{0} \times \frac{m}{m}$$

In the above equation, consider the limit for $\boldsymbol{n}$. In other words,

$$\lim_{n\to\infty} \frac{a}{0} \times \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{a \times 1/n}{0 \times 1/n} = \lim_{n\to\infty} \frac{a/n}{0} = \frac{0}{0} = 0$$

and

$$\lim_{m\to 0} \frac{a}{0} \times \frac{m}{m} = \lim_{m\to 0} \frac{a \times m}{0 \times m} = \lim_{m\to 0} \frac{am}{0} = \frac{0}{0} = 0$$

is obtained. However, the last equation uses 0/0 = 0. Here, if $1/\boldsymbol{n} = \boldsymbol{m}$, note that

$$\lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{m\to 0} \frac{m}{m} = 1$$

and

$$\lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{m\to 0} \frac{m}{m} \neq \frac{0}{0}$$

In other words, the limit system retains its form, and the ratio does not change even at the limit; however, in division by zero such as 0/0 = 0 where it is discontinuous to the limit, discrimination is necessary. In addition, the above result indicates that with $\boldsymbol{a}$ being any real number, dividing $\boldsymbol{a}$ with 0 approaches 0/0 = 0 infinitely. However, it should be noted that in a strict sense, there is a gap between the limit and a fixed point.

Reference: If $y = (x) = x$ is differentiated for $\boldsymbol{x}$,

$$y' = f'(x) = \lim_{\Delta x\to 0} \frac{(x + \Delta x) - x}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1$$

Therefore, the limit in the above division by zero is valid.