April 17, 2017
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The Density Principle Solved with a Graph, and Division by Zero

Theorem It may be beneficial to graph the density principle examined in the Division by Zero paper #39 ("The Density Principle and Division by Zero with the Resulting Remainder") and use the graph to demonstrate the principle. To this end, a coordinate system is considered with area along the x-axis and mass along the y-axis. In this coordinate system, let the maximum of the x-axis be s_0, which is the surface area of A_0, the bottom surface of a given cylinder. If some point along the x-axis between the origin and s_0 (the bottom surface area of the cylinder) is the target surface area s, the mass corresponding to that area, that is, the mass M_{in} sitting on top of the target surface A, is considered. The mass on top of A_0 is the total mass M. As the slope σ of the graph of the mass M_{in} with respect to the target surface area s is a function of s, it is expressed as $\sigma = \sigma(s)$. If now the cylinder has a uniform mass distribution ρ, then at every point from where the area is greater than 0 up to s_0 (the bottom surface area of the cylinder), the target surface area s and the partial mass atop it, that is, the mass M_{in} sitting on top of the target surface, is proportional with constant σ.

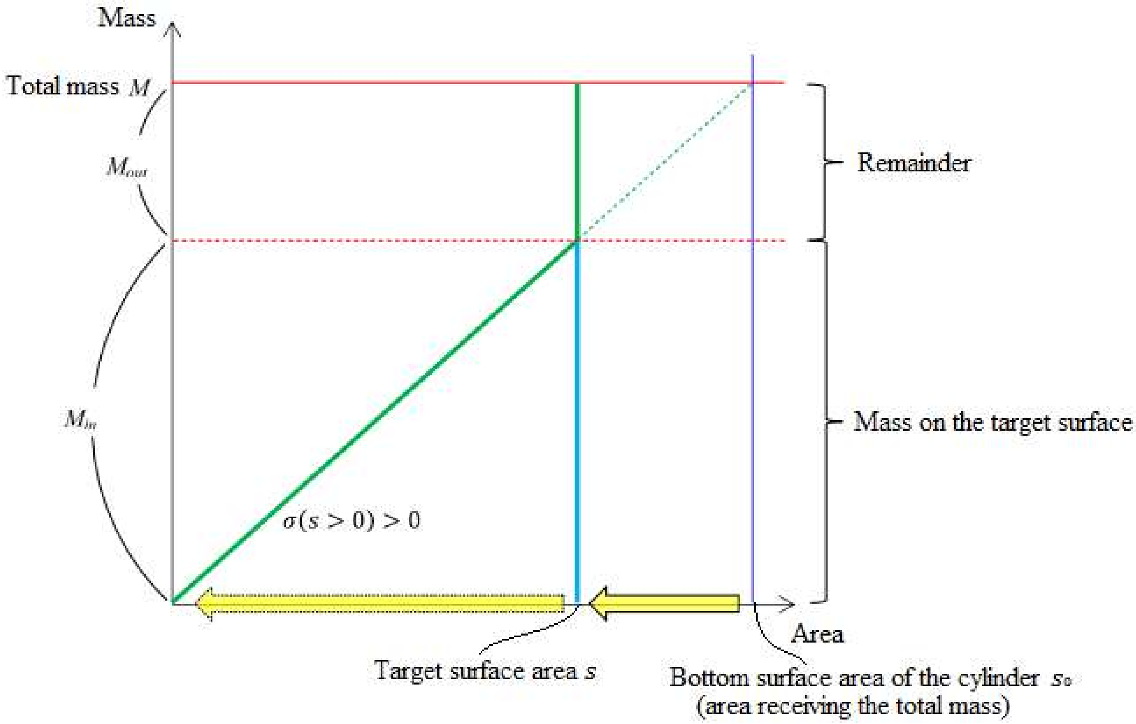

Fig 1. Density principle and division by zero: $\sigma(s)$ = const > 0

It is now assumed that starting from the state where the target surface area s is at its maximum, equal to the bottom surface area of the cylinder s_0, it is gradually shrunk. After shrinking it to a suitable extent, we obtain what is shown in Fig. 1, where the mass M_{in} sitting on the target surface area s can be determined by the linear equation $M_{in}=\sigma s$, and the total mass M can be determined by $M=\sigma s+ M_{out}$, a constant linear function of s.

To simplify the equations discussed so far, they will be reexamined by letting the bottom surface area of the cylinder $s_0 = 100$, the total mass $M = 100$, and the linear slope $\sigma = 1$ ($s > 0$). When the target surface area s is at its maximum, i.e., equal to the bottom surface area of the cylinder s_0, the ratio of the total mass M to the target surface area s, by way of compound numbers, is

$$\frac{100}{100}=\frac{100+0}{100}=1\frac{0}{100}=1+\frac{0}{100}=1+0=1 \qquad (1)$$

where the leading number 1 is the value of σ. Then, if s is shrunk by only 1/100,

$$\frac{100}{99}=\frac{99+1}{99}=1\frac{1}{99}=1+\frac{1}{99} \qquad (2)$$

Using compound numbers as in the previous expression, it can be seen that the ratio reaches a value slightly larger than 1. The leading number 1 is the value of σ as well.

As these formulations are rather counter-intuitive, the relations among the divided, divisor, quotient, and remainder will now be reorganized. Here, the initial numerator corresponds to the divisor, the denominator to the divisor, the leading number to the quotient, and the subsequent numerator to the remainder. The divisor does not change; thus, this relationship is expressed as

$$\frac{100}{100}=\frac{100+0}{100}=1\frac{0}{100} \Rightarrow 100=1\times 100+0$$

$$\frac{100}{99}=\frac{99+1}{99}=1\frac{1}{99} \Rightarrow 100=1\times 99+1$$

$$\frac{100}{98}=\frac{98+2}{98}=1\frac{2}{98} \Rightarrow 100=1\times 98+2$$

$$\frac{100}{97}=\frac{97+3}{97}=1\frac{3}{97} \Rightarrow 100=1\times 97+3$$

$$\vdots$$

$$\frac{100}{51}=\frac{51+49}{51}=1\frac{49}{51} \Rightarrow 100=1\times 51+49$$

$$\frac{100}{50}=\frac{50+50}{50}=1\frac{50}{50} \Rightarrow 100=1\times 50+50$$

$$\frac{100}{49}=\frac{49+51}{49}=1\frac{51}{49} \Rightarrow 100=1\times 49+51$$

$$\vdots$$

$$\frac{100}{3}=\frac{3+97}{3}=1\frac{97}{3} \Rightarrow 100=1\times 3+97$$

$$\frac{100}{2}=\frac{2+98}{2}=1\frac{98}{2} \Rightarrow 100=1\times 2+98$$

$$\frac{100}{1}=\frac{1+99}{1}=1\frac{99}{1} \Rightarrow 100=1\times 1+99$$

In this sequence of calculations, the first term in the expression following the arrow ⇒ corresponds to the size of the mass M_{in} sitting on the area, which is represented by the vertical blue line in Fig. 1 (parallel to the mass axis), and the second term corresponds to the mass M_{out} sitting outside the area, that is, the remainder, which is represented by the vertical green line (parallel to the mass axis).

Thus, if $s = 0$, then

$$\frac{100}{0} = \frac{0 + 100}{0} = 0\frac{100}{0} \Rightarrow 100 = 0 \times 0 + 100 \qquad (3)$$

As is clearly seen in Fig. 2, the target surface area s is 0; thus, M_{in}, the mass on it, is also 0. Moreover, M_{out}, the mass outside it, accounts for the entire total mass M and is thus 100.

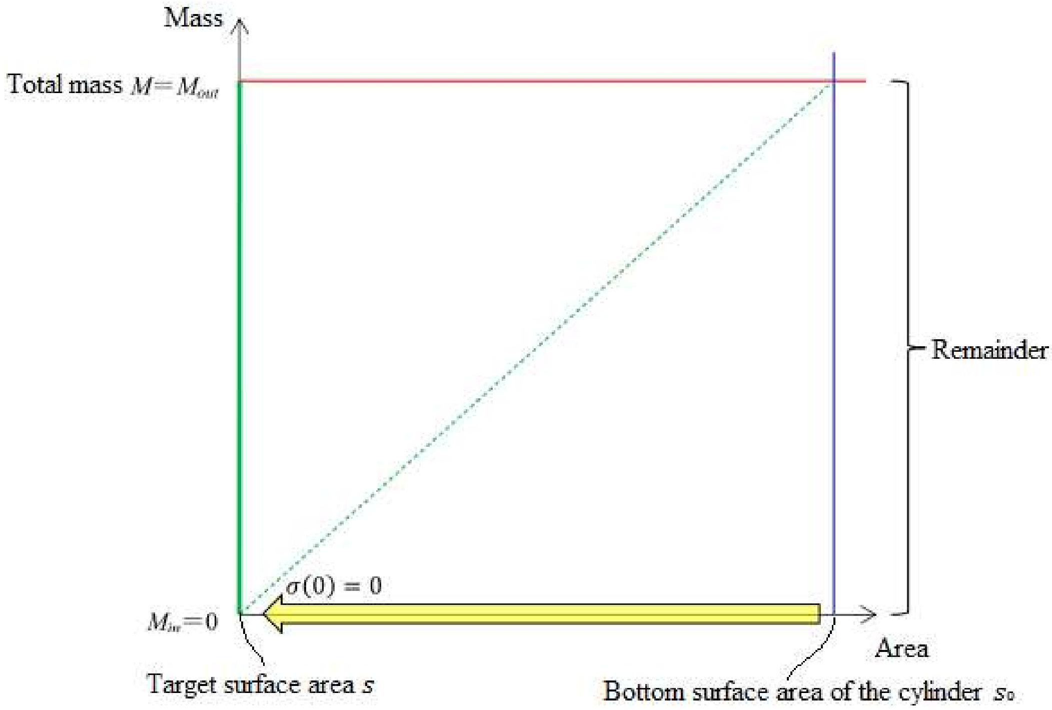

Fig. 2. Density principle and division by zero: $\sigma(s) = 0$

Furthermore, if the leading number is 1 in (3) as well, then

$$\frac{100}{0} = \frac{0 + 100}{0} = 1\frac{100}{0} = 1 + \frac{100}{0} = 1 + 1 + \frac{100}{0} = \infty + \frac{100}{0} = \infty\frac{100}{0}$$
$$\Rightarrow 100 = \infty \times 0 + 100 \qquad (4)$$

However, although $\infty \times 0$ implies that there is not even a single ∞, it is undefined as its magnitude is still unclear. That is, (4) is absurd, and the equation does not hold true. This will happen even if the leading number is changed to any real number other than 0. This shows that

$$\frac{0}{0} = 1 \qquad (5)$$

is absurd. Consequently, it is clear that $100 = \square \times 0 + 100$ is true if and only if $\square = 0$, yielding $100 = 0 \times 0 + 100$. Based on this, it is clear that

$$\frac{0}{0} = 0 \qquad (6)$$

What follows is an alternative proof for this.

For two real numbers $a, b \geq 0$, let

$$f(a,b) = f(b + (a - b), b) = f(b,b) + f(a - b, b) \qquad (7)$$

and

$$f(a,b) = \frac{a}{b} \quad (b \neq 0) \qquad (8)$$

Then,

$$f(a,0) = f(0 + (a - 0), 0) = f(0,0) + f(a - 0,0) = f(0,0) + f(a,0)$$
$$\therefore f(0,0) = 0 \qquad (9)$$

Hence, (6) is obtained:

$$\frac{0}{0} = 0$$

Furthermore, from the above hypothesis, the following holds for two real numbers a, $b \geq 0 \land a \geq nb$ ($n \in \mathrm{N}$):

$$\begin{aligned} f(a,b) &= f(b + (a - b), b) = f(b,b) + f(a - b, b) = f(b,b) + f(b + \{(a - b) - b\}, b) \\ &= f(b,b) + f(b,b) + f\big((a - b) - b, b\big) = 2f(b,b) + f(a - 2b, b) = \cdots \\ &= nf(b,b) + f(a - nb, b) \quad (a \geq nb) \qquad (10) \end{aligned}$$

Thus, if $b = 0$, then

$$f(a,0) = nf(0,0) + f(a - n \times 0,0) = nf(0,0) + f(a,0) \quad (a \geq nb)$$
$$\therefore f(0,0) = 0 \qquad (11)$$

This shows that when (3) is written in the form of a compound fraction,

$$\frac{\text{dividend } 100}{\text{divisor } 0} = \frac{\text{converted dividend[A1] } 0 + \text{remainder } 100}{\text{divisor } 0} = \text{quotient } 0 \ \frac{\text{remainder } 100}{\text{divisor } 0}$$

$\Rightarrow$ dividend 100 = quotient 0 divisor 0 + remainder 100 (3')

the quotient is uniquely determined to be 0, based on (6). Consequently, (3) holds true. Generally, the following is true:

$\frac{\text{dividend } a}{\text{divisor } 0} = \text{quotient } 0 \cdots \text{remainder } a \land \text{dividend } a = \text{quotient } 0 \times \text{divisor } 0 + \text{remainder } a$
(12)

It should be noted that the converted dividend in (3′) is the dividend obtained after the original dividend splits into the sum of a secondary dividend and the remainder. When the divisor is not 0, the dividend can be split arbitrarily; however, when the divisor is 0, the only way to split the dividend so that the number can be turned into a compound fraction results in a converted dividend of 0. That is, when the divisor is 0, the quotient is uniquely 0, and in terms of reducible set theory, the remainder is uniquely equal to the dividend. This may be redundant, but if $b = 0$ when $a \neq 0$, then $x \neq 0$, and thus

$$\frac{a}{0} = \frac{a}{0} \times 1 = \frac{a}{0} \times \frac{x/a}{x/a} = \frac{x}{0} \qquad (a, x \neq 0) \qquad (13)$$

Considering $x = 1$ yields

$$\frac{a}{0} = \frac{1}{0} \qquad (a \neq 0) \qquad (14)$$

This shows that a ratio (comparison) with 0 only makes it possible to compare the two qualitative states of “being” and “non-being.” This shows that a given size cannot be quantitatively measured using the size itself as reference, because the dividend, which is the reference size, is 0. Furthermore, when there is a difference in the size of the numerator, no conclusion can be drawn. Nevertheless, it does show that there is a state of “being,” which is distinct from the state of “non-being.”

The notation of compound numbers is as follows. Fractional equalities will be denoted using three real numbers a, b, as

$$\frac{ab + c}{b} = a\frac{c}{b} = a + \frac{c}{b} \equiv a\_\frac{c}{b} \qquad (15)$$

and a is called the leading number, b the divisor, ab+c the dividend, and c the remainder. Then, if the leading number a is called the general quotient, the general quotient a that minimizes the remainder c is equivalent to the quotient A of the previous division. That is,

$$a_{max\frac{c_{min}}{b}} \Rightarrow (ab + c) \div b = A \cdots C \qquad (16)$$

in which case

$$a_{max} = A \wedge c_{min} = C \qquad (17)$$

and the following relation between the set $\{a\}$ of general quotients a and quotient A, as well as between the set $\{c\}$ of remainders c and C, is true:

$$A \in \{a\}, C \in \{c\} \qquad (18)$$

Using the above notation for compound numbers, (7) can be written as

$$f(a, b) = f(b + (a - b), b) = f(b, b) + f(a - b, b) = f(b, b)\_f(a - b, b) \qquad (7')$$

Then, Equation (8) will be established

$$f(a, b) = \frac{a}{b} \quad (b \neq 0)$$

whereupon if $b = 0$,

$$f(a, 0) = f(0,0)\_f(a, 0) \qquad (19)$$

and thus (9)

$$f(0,0) = 0$$

holds true. In this case, by (9), Equation (19) is uniquely expressed as

$$f(a,0) = 0\_f(a,0) \qquad (20)$$

Moreover, by the definition of compound numbers, the leading number 0 in (20) is an element in the set of general quotients; moreover, it is the largest in the set, as (20) is uniquely true. It is also equivalent to the quotient of the previous division. Consequently,

$$\frac{a}{0} = 0\frac{a}{0} = 0 \cdots a \qquad (21)$$

is uniquely true.

A coordinate system with area along the x-axis and mass along the y-axis will again be considered. In this coordinate system, let the maximum of the x-axis be s_0, which is the surface area of A_0, the bottom surface of a given cylinder. Let A be the pressure receptor of a flat and perfectly rigid body, with a surface area s at some point along the x-axis between the origin and s_0 (the bottom surface area of the cylinder) such that A receives the total mass M_{on} of a cylinder with a uniform base. That is, M_{on}, the total mass acting on pressure receptor A, is considered. Of course, the mass on the underside of the cylinder A_0 is equal to the mass M. Then, the slope σ of the graph of the mass M_{on} with respect to the surface area of the pressure receptor s is expressed as a function of s, $\sigma = \sigma(s)$. Now, if the cylinder has a uniform mass distribution ρ and is a perfectly rigid body, then at every point from where the area is greater than 0 up to s_0 (the bottom surface area of the cylinder), the target surface area s and the total mass M of the cylinder sitting atop it (the mass M_{on} acting on the pressure receptor) are in a linear relationship with the size of $\sigma(s)$, that is, the slope of the graph σ, as it changes.

It is now assumed that starting from the state where the surface area of the pressure receptor s is at its maximum, equal to the bottom surface area of the cylinder s_0, s gradually shrinks. After it shrinks to a suitable extent, we obtain what is shown in Fig. 3, where the mass M_{on} on the surface area of the pressure receptor s remains constant and can be found by the linear equation $M_{on} = \sigma(s)s$. That is, $\sigma(s)$ is inversely proportional to s. Of course, the total mass M is expressed as the sum of M_{on}, the mass on the surface area of the pressure receptor A, and M_{off}, the mass not on the surface area of the pressure receptor, yielding

$$M = M_{on} + M_{off} \qquad (22)$$

This implies that the total mass M can be expressed as

$$M = \sigma(s)s + M_{off} \qquad (23)$$

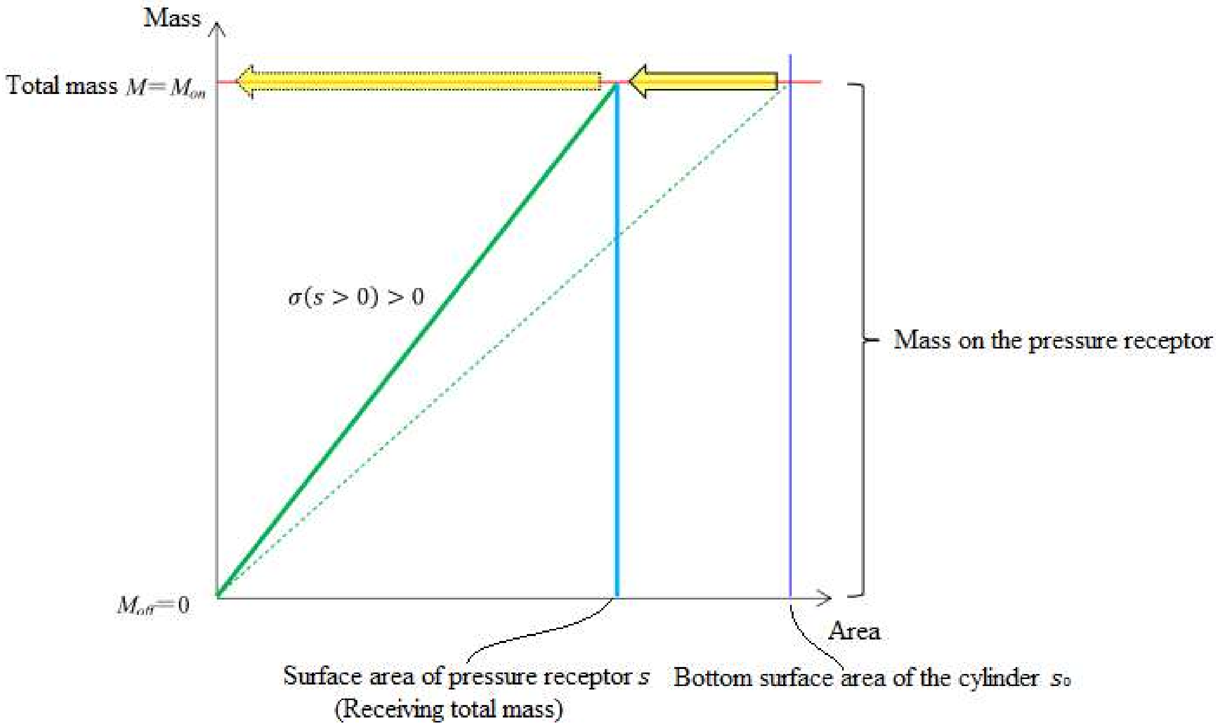

Fig. 3. Density principle and division by zero: [MATH]

To simplify the equations discussed so far, they will be reexamined by letting the bottom surface area of the cylinder s_0= 100, the total mass M = 100, and the linear slope $\sigma(s) \geqq 1$ ($s > 0$). When the target surface area s is equal to the bottom surface area of the cylinder s_0, to determine the ratio of the total mass M to the surface area of the pressure receptor s, if the dividend is split into the sum of two numbers to minimize the remainder, it can be seen that, by way of compound numbers, it is

$$\frac{100}{100} = \frac{100+0}{100} = 1\frac{0}{100} = 1 + \frac{0}{100} = 1 + 0 = 1 \qquad (24)$$

where the leading number 1 is the value of σ. Then, if s is shrunk by only 1/100,

$$\frac{100}{99} = \frac{100+0}{99} = 1.\dot{0}\dot{1}\frac{0}{99} = 1.\dot{0}\dot{1} + \frac{0}{99} = 1.\dot{0}\dot{1} \qquad (25)$$

Using compound numbers as in the previous expression, it can be seen that the ratio reaches a value slightly larger than 1. The leading number $1.\dot{0}\dot{1}$ is here the value of σ as well.

As before, the relations among the divided, divisor, quotient, and remainder below will be organized. Here, the initial numerator corresponds to the divisor, the denominator to the divisor, the leading number to the quotient, and the subsequent numerator to the remainder. The divisor does not change; thus, this relationship is expressed as

$$\frac{100}{100}=\frac{100+0}{100}=1\frac{0}{100} \qquad \Rightarrow\ 100=1\times 100+0$$

$$\frac{100}{99}=\frac{100+0}{99}=\dot{1}.0\dot{1}\frac{0}{99} \qquad \Rightarrow\ 100=\dot{1}.0\dot{1}\times 99+0$$

$$\frac{100}{98}=\frac{100+0}{98}=1.02\cdots\frac{0}{98} \qquad \Rightarrow\ 100=1.02\cdots\times 98+0$$

$$\frac{100}{97}=\frac{100+0}{97}=1.02\cdots\frac{0}{97} \qquad \Rightarrow\ 100=1.03\cdots\times 97+0$$

$$\vdots$$

$$\frac{100}{51}=\frac{100+0}{51}=1.96\cdots\frac{0}{51} \qquad \Rightarrow\ 100=1.96\cdots\times 51+0$$

$$\frac{100}{50}=\frac{100+0}{50}=2\frac{0}{50} \qquad \Rightarrow\ 100=2\times 50+0$$

$$\frac{100}{49}=\frac{100+0}{49}=2.04\cdots\frac{0}{49} \qquad \Rightarrow\ 100=2.04\cdots\times 50+0$$

$$\vdots$$

$$\frac{100}{3}=\frac{100+0}{3}=33.\dot{3}\frac{0}{3} \qquad \Rightarrow\ 100=33.\dot{3}\times 3+0$$

$$\frac{100}{2}=\frac{100+0}{2}=50\frac{0}{2} \qquad \Rightarrow\ 100=50\times 2+0$$

$$\frac{100}{1}=\frac{100+0}{1}=100\frac{0}{1} \qquad \Rightarrow\ 100=100\times 1+0$$

According to the density principle 0/0 = 0 as well as (21),

$$\frac{100}{0}=\frac{0+100}{0}=0\frac{100}{0}\ \Rightarrow\ 100=0\times 0+100 \qquad (26)$$

is uniquely true. This also takes the exact same form as (3), which expresses Fig. 2 numerically. The graph in Fig. 4 is similarly equivalent to Fig. 2 as well. The only difference between these two systems is that the former has a surface outside the target to support the mass sitting outside it, whereas the latter has no surface outside the pressure receptor to support the mass sitting outside it. This implies that in the former, the cylinder does not undergo any changes even when the target surface area is 0, whereas in the latter, when the surface area of the pressure receptor is 0, there ceases to be any surface capable of supporting any of the cylinder’s mass, causing the cylinder to fall.

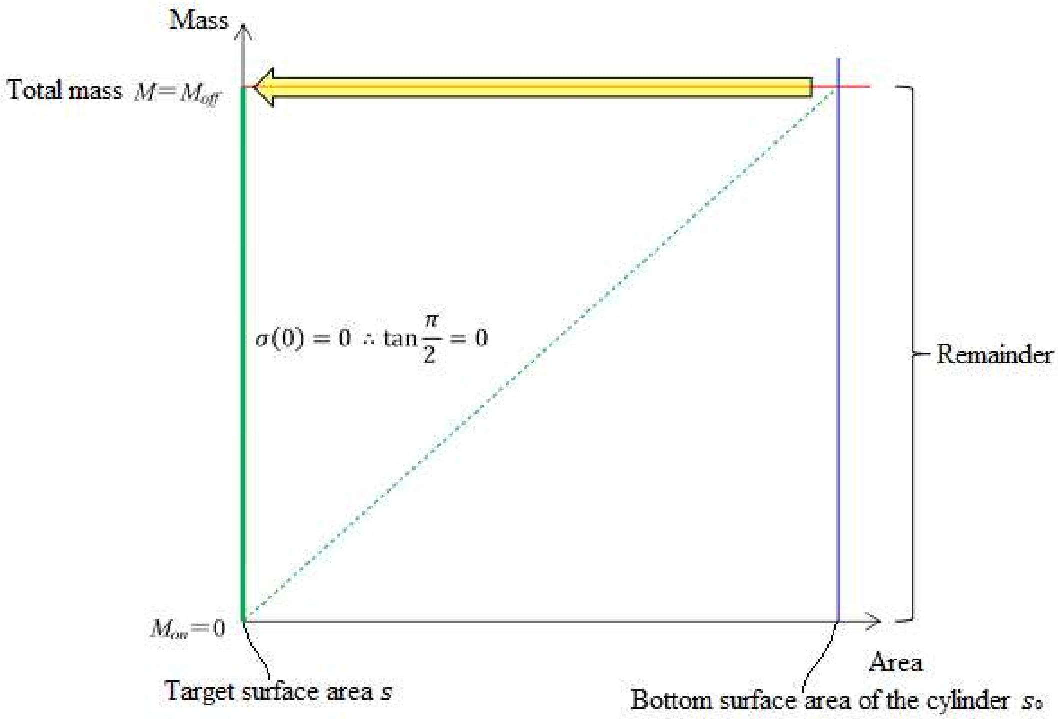

Fig. 4. Density principle and division by zero: $\sigma(s)=0$

So far, the density principle has been explained in terms of ratios and division by zero, using mass surface density as an example. This could be called the ratio principle. Some examples will now be used to deepen the understanding of the density principle (ratio principle) as seen in Figs. 3 and 4. The density principle will be first considered followed by the ratio principle. For the former, a disappearing island is considered. This island has a total area of 100 ha and is home to 100 residents. There are no births or deaths, nor is there immigration from other areas. However, there is one problem: owing to global warming and sea level rise, the area of this island decreases by 1 ha every year.

The current population density is 1 person/ha; in 20 years, it will be 1.25 people/ha. What will it be in 100 years?

Current population density:

$$\frac{100[\text{people}]}{100[\text{ha}]} = 1[\text{people/ha}]\,\frac{0[\text{people}]}{100[\text{ha}]} = 1[\text{people/ha}] \text{ remainder } 0 \text{ [people]}$$

Population density in 20 years:

$$\frac{100[\text{people}]}{100 - 1 \times 20[\text{ha}]} = \frac{100[\text{people}]}{80[\text{ha}]} = 1.25[\text{people/ha}]\,\frac{0[\text{people}]}{80[\text{ha}]} = 1.25[\text{people/ha}] \text{ remainder } 0 \text{ [people]}$$

Population density in 50 years:

$$\frac{100[\text{people}]}{100 - 1 \times 50[\text{ha}]} = \frac{100[\text{people}]}{50[\text{ha}]} = 2[\text{people/ha}]\,\frac{0[\text{people}]}{50[\text{ha}]} = 2[\text{people/ha}] \text{ remainder } 0 \text{ [people]}$$

Population density in 90 years:

$$\frac{100[\text{people}]}{100 - 1 \times 90[\text{ha}]} = \frac{100[\text{people}]}{10[\text{ha}]} = 10[\text{people/ha}]\,\frac{0[\text{people}]}{10[\text{ha}]} = 10[\text{people/ha}] \text{ remainder } 0 \text{ [people]}$$

Population density in 99 years:

$$\frac{100[\text{people}]}{100 - 1 \times 99[\text{ha}]} = \frac{100[\text{people}]}{1[\text{ha}]} = 100[\text{people/ha}]\ \frac{0[\text{people}]}{1[\text{ha}]} = 100[\text{people/ha}] \text{ remainder } 0\ [\text{people}]$$

That is, 100 people on 1 ha of land. What will happen 1 year after that? After 100 years, the island will be completely submerged; hence, its surface area will be 0. However, as there are no births, deaths, or immigration, the residents will have become sea-people floating in the sea. This is the same as trying to determine the population density if there were no island to begin with. That is, as there is no "island population density" in the water, where there is no island, the population density is null φ whether people are there or not (in the sense of the population density of the island). This is the same as being zero. Thus, the case where there is no island to begin with is mathematically equivalent to the case where the island disappears. Consequently,

Population density in 100 years:

$$\frac{100[\text{people}]}{100 - 1 \times 100[\text{ha}]} = \frac{100[\text{people}]}{0[\text{ha}]} = 0[\text{people/ha}]\ \frac{100[\text{people}]}{0[\text{ha}]} = 0[\text{people/ha}] \text{ remainder } 100\ [\text{people}]$$

Moreover, for the ratio principle, the relationship between the mass and moving velocity of a neutrino will be examined. Let the *y*-axis be the rest mass of the particle m_0, and the *x*-axis be the inverse of the Lorentz factor corresponding to the particle's velocity v, that is,

$$\sqrt{1 - \frac{v^2}{c^2}}$$

(which will hereafter be referred to as the Lorentz quantity). Consequently, if the velocity v is assigned a value within the range $0 \leq v \leq c$, where c is the speed of light, the value of the *x*-axis will be between 0 and 1. If it is assumed that this graph passes through the origin, its slope represents (v), the mass of the particle at velocity v, as Fig. 5 shows. That is,

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

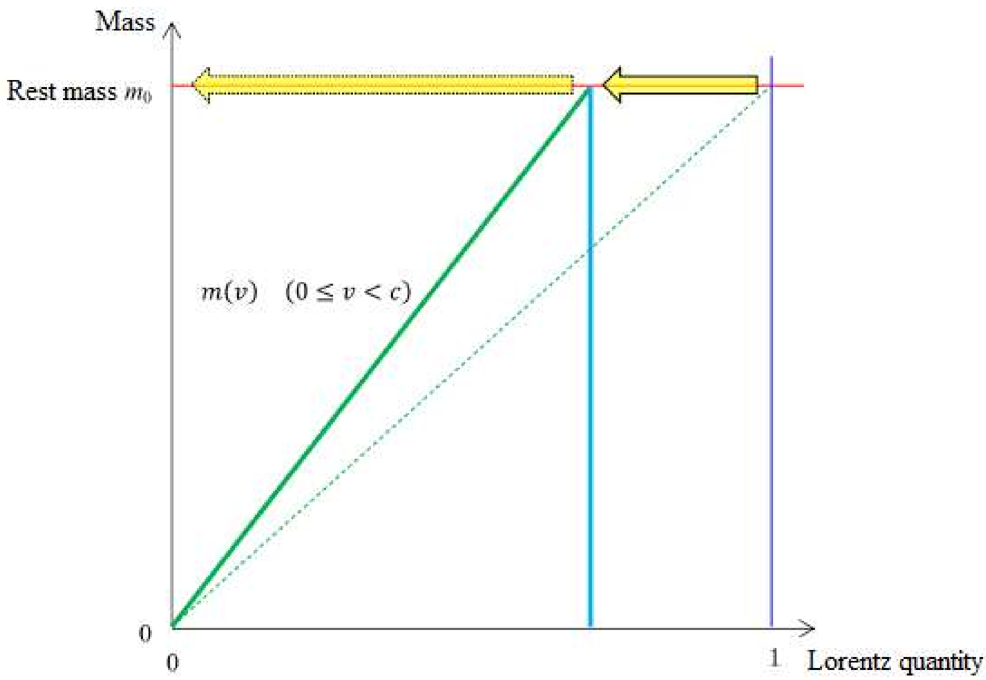

Fig. 5. Relationship between a neutrino's velocity v and its mass (v) $(0 \leq v < c)$

When the particle's velocity v is equal to the speed of light c, the Lorentz quantity is 0, the graph is equal to the y-axis, and its height is equal to the rest mass m_0. What is (c), the mass of the particle when its velocity v is the speed of light c?

For particle velocity v in the range of $0 \leq v < c$,

$$\text{relativistic mass } m(v) = \frac{\text{non}-\text{relativistic mass } m_0}{\text{Lorentz quantity}\sqrt{1-\frac{v^2}{c^2}}}$$
$$= \frac{\text{non}-\text{relativistic mass } m_0 + \text{non}-\text{relativistic mass } 0}{\text{Lorentz quantity}\sqrt{1-\frac{v^2}{c^2}}}$$
$$= \text{non}-\text{relativistic mass } m(v)\_\frac{\text{non}-\text{relativistic mass } 0}{\text{Lorentz quantity}\sqrt{1-\frac{v^2}{c^2}}}$$
$$= \text{non}-\text{relativistic mass } m(v) \text{ remainder non}-\text{relativistic mass } 0$$

Therefore, if v=c,

$$\text{relativistic mass } m(c) = \frac{\text{non}-\text{relativistic mass } m_0}{\text{Lorentz quantity}\sqrt{1-\frac{c^2}{c^2}}}$$
$$= \frac{\text{non}-\text{relativistic mass } 0 + \text{non}-\text{relativistic mass } m_0}{\text{Lorentz quantity } 0}$$
$$= \text{relativistic mass } 0\_\frac{\text{non}-\text{relativistic mass } m_0}{\text{Lorentz quantity } 0}$$
$$= \text{non}-\text{relativistic mass } m(c) = 0 \text{ remainder non}-\text{relativistic mass } m_0$$

Consequently, the increase in mass (c) of a particle due to the relativistic effect disappears when the particle moves at the speed of light c, that is, its relativistic mass (c) is 0, and its rest mass m_0 is measured as the non-relativistic mass.