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Expansion of Congruence Statements and Division by Zero

Theorem For a, $b \in Z$, the following is true:

$$a \equiv b(\mathrm{mod}.0) \Rightarrow a = b$$

Proof $0|a$ is true if and only if $a = 0$ because by the fundamental theorem of reducible set-theoretical division by zero shown in the following expression:

$$\frac{a}{0} = 0 \cdots a \qquad (1)$$

(where $\cdots a$ denotes the remainder of a), $\cdots a = 0$ is true if and only if $a = 0$. Consequently, when $a = 0$, the statement in (1) is true if and only if $0|b$, that is, $b = 0$. In this case, $a = b = 0$, and thus

$$0 \equiv 0(\mathrm{mod}.0) \Rightarrow a = b \qquad (2)$$

When $a \neq 0$, this satisfies $\cdots a \neq 0$ in (1). Moreover,

$$\frac{b}{0} = 0 \cdots b \qquad (3)$$

and thus the statement is true if and only if $\cdots b = a$; that is,

$$\cdots a = \cdots b \qquad (4)$$

Of course, (4) satisfies (1) and (3); therefore,

$$\cdots a = \cdots b \Rightarrow a = b \qquad (5)$$

Conversely, when the modulo is 0, if $a \neq b$, then

$$a \not\equiv b(\mathrm{mod}.0) \qquad (6)$$

$$\because \quad \frac{a-b}{0} = 0 \cdots a - b \neq 0$$

$$\therefore a \equiv b(\mathrm{mod}.0) \Rightarrow a = b$$

QED.