August 20, 2014
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Proof that argument for irrationality of division by zero observed in multiplication as an inverse operation of division is incorrect

Because the relationship

$$\frac{a}{b} = c \Rightarrow a = bc \qquad (1)$$

holds, where a, b, $c \in \mathrm{R}$, in a method (proof) that shows the irrationality of division by zero, when $b = 0$ where $a \neq 0$, we obtain $a = 0$, which is irrational. Therefore, division by zero, or $a/0$, is not defined. This specifically states the irrationality of:

$$\frac{a}{0} = c \Rightarrow a = 0 \times c \quad (a \neq 0) \qquad (2)$$

(Here, the result of $0 \times c$ seen on the right side is 0. Note that it is easy to show that there is no real number that does not become 0 when multiplied by 0. We will show this in the lemma below.)

However, this argument is not only incomplete as a proof, but also incorrect. We will show this below.

Theorem: The relationship between division and multiplication as inverse operations cannot prove the irrationality of division by zero.
Proof: For the relationship in Equation (1) to hold, the following assumption is necessary. In other words, whether the relationship in Equation (1) can hold depends on the following.

Both sides of Equation (3) are multiplied by $\boldsymbol{b}$:

$$\frac{a}{b} = c \qquad (3)$$

Which results in

$$b \times \frac{a}{b} = b \times c \qquad (4)$$

The left side of Equation (4) becomes

$$b \times \frac{a}{b} = a \times \frac{b}{b} = a \times 1 \qquad (5)$$

In other words, it is a question of whether

$$\frac{b}{b} = 1 \qquad (6)$$

generally holds or not, meaning the derivation of irrationality of division by zero in the relationship of Equation (1) above assumes that 0/0 = 1 holds.

However, whether 0/0 = 1 holds or not must be proven. Therefore, whether Equation (1) generally holds or not depends on whether 0/0 = 1 holds or not. Thus, the above irrationality occurring in the relationship of Equation (1) alone cannot deny division by zero.

Thus, if we assume 0/0 = 1, and $A = B \in \mathrm{R}$, and multiply both sides with B

$$AB = B^2 \qquad (7)$$

and then subtract A^2 from both sides

$$AB - A^2 = B^2 - A^2 \qquad (8)$$

This is rearranged to obtain

$$A(A - B) = (A + B)(A - B) \qquad (9)$$

Then, the assumption clearly allows for division of both sides by $(A - B)$, and because that leads to

$$A \times \frac{A - B}{A - B} = (A + B) \times \frac{A - B}{A - B} \qquad (10)$$

assuming $A = B$, it is immediately reduced to

$$A \times \frac{0}{0} = 2A \times \frac{0}{0} \qquad (11)$$

Therefore, because A is an arbitrary real number, if $A \neq 0$,

$$1 = 2 \ (12)$$

If $A = 0$, based on the assumption, Equation (11) becomes

$$A = 2A \ (13)$$

and if both sides of this equation are divided by 0,

$$\frac{0}{0} = 2\frac{0}{0} \qquad (14)$$

Therefore,

$$1 = 2 \times 1 \ (15)$$

Therefore

$$1 = 2 \quad (16)$$

Equations (12) and (16) are both irrational. The reason such irrationality occurred is because 0/0 = 1 was assumed to hold. Therefore, 0/0 = 1 does not hold.
As such, it was proven that it is impossible to show that division by zero is irrational in the relationship of Equation (1).

By this theorem, it was shown that it is impossible to show the irrationality of division by zero in the relationship of division and multiplication as its inverse operation. However, this theorem shows that defining division as an inverse operation of multiplication is a narrow definition of division. In other words, such a definition of division is limited to cases where the relationship $b/b = 1$ holds for the divisor $\boldsymbol{b}$ in division a/b. Division with such a limiting condition is indeed a narrow definition of division.

Lemma: There is no real number which does not become 0 (real number 0) when multiplied by 0 (real number 0).
Proof: Let us assume that there is a special value 0 that does not become 0 if multiplied by 0, but can be applied for general numbers in processes such as the four basic arithmetic operations, distributive property, and associative law. At this time, if we multiply both sides of

$$0 = 0 + 0$$

By 0, we obtain

$$0 \times 0 = (0 + 0) \times 0$$

$$= 0 \times 0 + 0 \times 0$$

Here, if we use notation $0 \times 0 = \mu \neq 0$, the above equation becomes

$$\mu = 2\mu$$

Thus,

$$\mu = 0$$

is obtained. However, this is inconsistent with the assumption. In other words, for any real number x, the solution of multiplying by 0 is 0; meaning, it must be

$$x \times 0 = 0$$

and $a = 0$ is no exception. Of course, in the case of 0×0, it must be

$$0 \times 0 = 0$$